# ZECHYP: A NUMERICAL EVALUATOR OF REAL ZEROS OF THE CONFLUENT HYPER-GEOMETRIC FUNCTION FOR REAL ARGUMENTS

*Dr. J.J.S. Bisht*

The smallest positive zeros of the confluent hyper geometric function of first kind were first calculated by Slater on EDSAC – I computer in late 1960 and tabulated up to seven places of decimals. In the present paper the author has tried to develop an algorithm to compute these real zeros and based on this algorithm the author has further attempted to compute and tabulate next higher zeros of CHF function for a variety of test cases with greater accuracy and efficacy. The source code is written in the C language and the program is running successfully under the Linux/UNIX environment and giving precise output results.

**Keywords:** Confluent hyper geometric function,Evaluator,Bessel functions, Coulomb functions, Hankel functions, Kummer's function, Newton's approximation.

## 1.INTRODUCTION

It has been proved in literature that every confluent hyper geometric function (CHF) ${}_1F_1[a;b;x]$ has at most only a finite number of real zeros for any fixed values of a and b [1]. Some more detailed investigation of the number of real zeros of ${}_1F_1[a;b;x]$ when a and b are real, has been done in [2].

The confluent hyper-geometric function is a solution of the differential equation,

$$Z\,{}_1F_1''(a;b;Z) + (b - Z)\,{}_1F_1'(a;b;Z) - a\,{}_1F_1(a;b;Z) = 0 \qquad (1.1)$$

Where a, b, and Z may all be complex. An exact solution of this equation is given by Kummer's series [3]. A well - known property of analytic function is that it can be expressed as an infinite product in terms of its zeros [4]. Since CHF function ${}_1F_1[a;b;x]$ is an analytic function of x except when b is zero or a negative integer, we can write

$${}_1F_1[a;b;x] = e^{ax/b}\prod_{n=0}^{\infty}[1 - x/x_n]e^{x/xn} \qquad (1.2)$$

where $x_0$, $x_1$,…………., $x_n$ are the real and imaginary zeros of the function.

For n = 0,

$$ {}_1F_1[a;b;x] \approx \left[1 - \frac{x}{x_0}\right] \exp\left[\frac{ax}{b} + \frac{x}{x_0}\right] \qquad (1.3)$$

Where, $x_0$ is the real zero nearest to the plane x = 0. This provide us a first approximation to the numerical value of ${}_1F_1[a;b;x]$ for real values or complex values of x near $x_0$. Thus having found the zeros we can construct the function numerically.

## 2. APPROXIMATION TO THE ZEROS

From the expansion of Kummer's function in terms of Bessel functions[5], Tricomi[6] has deduced that if Xr is the rth positive zero of ${}_1F_1[a;b;x]$ and $j_{b-1,r}$ is the rth positive zero of the Bessel function $J_{b-1}(x)$ then, for k ≡ ½b – a large,

$$ X_r = \frac{j_{b-1,r}^2}{4k}\left[1 + \frac{2b(b-2) + j_{b-1,r}^2}{48k^2}\right] + o(k^{-4}) \qquad (2.1)$$

Using the approximation for $j_{b-1,r}$ given by Watson[4], Slater[5] has deduced a very useful approximation for $X_r$ and found that

$$ J_{b-1,r} \approx Л( r + ½b - ¾) \qquad (2.2)$$

Therefore,

$$ X_r \approx \frac{Л^2( r + ½b - ¾)^2}{2b – 4a} \qquad (2.3)$$

The above approximation is sufficiently accurate to provide a starting-point for the nesting process described in the next section, even in the case of the smallest positive zero $X_0$.

## 3. NESTING PROCESSES

The rough evaluation of the zeros of CHF function is possible with the help of an ordinary calculating machine, but, if high accuracy is required and an electronic computer is available, it is more profitable to develop programs based on the process described below:

For the solution of CHF function let us calculate an initial estimate of the zero k by (2.3) and using Newton's approximation, we have

$$X_1 = k - y(k)/y'(k) \tag{3.1}$$

Where, $y'(k) = \dfrac{a}{b}\, {}_1F_1[a+1;b+1;k]$

Similarly, we have

$$X_1 = X_0 - \frac{b\ {}_1F_1[a;b;X_0]}{a\ {}_1F_1[a+1;b+1;X_0]} \tag{3.2}$$

and the cycle proceeds until,

$$X_{n+1} = X_n - \frac{b\ {}_1F_1[a;b;X_n]}{a\ {}_1F_1[a+1;b+1;X_n]} \tag{3.3}$$

## 4. EVALUATION OF ZEROS

The recurrence relation (3.3) includes evaluation of CHF function, so to evaluate this function a direct summation of Kummer's Series by taking finite number of terms is presented. In originally testing the program for CHF function, the results were compared against tables given for real values of arguments [3,5]. The Coulomb wave function and the Bessel function may also be evaluated using tables, along with the Hankel function [3,7,14]. The most comprehensive program we could find on this subject still had regions of difficulty where an answer could not be obtained [8]. Nardin [10, 11] gave a numerical evaluator for the confluent hypergoemetric function for complex arguments with large magnitudes. Recently an evaluator for the generalized hyper-geometric series has been presented by the same group of authors [12]. The wronskian of a function is, in general, considered to be of great use in checking tables of function [13]. The smallest positive zeros ( for r = 1) of CHF function has been tabulated by Slater [1, 5].

## 5. SAMPLE DATA AND PROGRAMS

Here the next ( higher surfaced ) zeros have been evaluated by taking**,**

1) r = 2, 3, 4, 5, and 6;
2) a = -4.0(0.1)-0.1;
3) b = 0.1(0.1)2.5;

The computer program (written in C language) and output results have been tabulated by Bisht [15] in Appendix -I and Appendix -II.

## 6. TIME REQUIREMENTS

Since for the evaluation of a single zero of CHF, the ${}_1F_1$ function has to be approximated two times in a single step and for the evaluation of ${}_1F_1$ function we are looping through enough number of terms for the CHF to converge, the time required to obtain a zero value can sometimes be large. For r = 2, a = -0.1, b = 0.1 the evaluator takes 10 seconds. Timings for the evaluator roughly ranged from several milliseconds to 10-20 seconds. This time may vary from system to system for a range of test cases.

## 7. USEFULNESS

It is pertinent to remark here that my supervisor Prof. J.M.C. Joshi, the Ex- Dean faculty of science, D.S.B. Campus, Kumaun University Nainital published a research paper in a foreign journal or reviews [16] and shown that CHF function has been used in statistics, so evaluation of its zeros numerically may be more useful there. S.M. Joshi Transform and SMPJ formula [17] (so called by the author in dedication to his parents) has used confluent hyper geometric function in its kernel and formula respectively. The usefulness of the CHF function has also been indicated in the book [18].

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***By- Dr. J.J.S. Bisht***
***Scientist – D***
***Dept. of Electronics & Information Technology (DeitY)***
***National Informatics Center (NIC)***
***Uttarakhand State Unit,***
***Secretariat,***
***Subhash Road,***
***Dehradun-248001 (INDIA)***
***E-mail: jjs.bisht@nic.in***, *jjbisht@gmail.com*